

On the geometrization of matter by exotic smoothness

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Abstract In this paper we discuss the question how matter may emerge from space. For that purpose we consider the smoothness structure of spacetime as underlying structure for a geometrical model of matter. For a large class of compact 4-manifolds, the elliptic surfaces, one is able to apply the knot surgery of Fintushel and Stern to change the smoothness structure. The influence of this surgery to the Einstein-Hilbert action is discussed. Using the Weierstrass representation, we are able to show that the knotted torus used in knot surgery is represented by a spinor fulfilling the Dirac equation and leading to a mass-less Dirac term in the Einstein-Hilbert action. For sufficient complicated links and knots, there are "connecting tubes" (graph manifolds, torus bundles) which introduce an action term of a gauge field. Both terms are genuinely geometrical and characterized by the mean curvature of the components. We also discuss the gauge group of the theory to be $U(1) \times SU(2) \times SU(3)$.

Keywords Fintushel-Stern knot surgery; K3 surface; spinor and gauge field by exotic smoothness

1 Introduction: A geometrical model of matter

The proposal to derive matter from space was considered by Clifford as well by Einstein, Eddington, Schrödinger and Wheeler with only partly success (see [46]). In a recent overview, Giulini [33] discussed the status of geometrodynamics in establishing particle properties like spin from spacetime by using special solutions of general relativity. Similar ideas are discussed in the model

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of Bilson-Thompson [12], in its loop theoretic extension [13] and in a model of Finkelstein [23] using the representation of knots by quantum groups. These approaches are using generalized geometric structures, special solutions of general relativity or a larger class of connections in some vector bundles. In this paper we will consider the smoothness structure of a 4-manifold as the underlying structure of the model. Only manifolds of the dimension four have an infinity of possible non-diffeomorphic smoothness structures (see for instance [35, 54, 24]). For all other dimensions, there are only finite many smoothness structures [39, 40]. This richness of the smoothness structures in four dimensions should have a physical meaning. There is a growing literature discussing the influence of exotic smoothness on physics. It is a common belief that exotic smoothness is a main contribution to the path integral of quantum gravity which was confirmed in a special case by one of the authors [6]. The topic started with the paper of Brans and Randall [16] and later by Brans [15, 14] leading to the guess that exotic smoothness can be a source of non-standard solutions of Einstein's equation (Brans conjecture). One of the authors published an article [5] to show the influence of the differential structure to GRT for compact manifolds of simple type. Śladowski showed in [53] that the exotic \mathbb{R}^4 can act as the source of the gravitational field. Thus exotic smoothness is able to represent a source of a gravitational field which cannot be distinguished from a usual source by an external observer. Furthermore, these sources are localized in the 4-manifold, i.e. one can construct a non-diffeomorphic smoothness structure from a given one by a modification of a 4-dimensional submanifold. As far as we know, sources of a gravitational field are any kind of matter (baryonic, radiation or dark). From all this it seems natural to relate matter with exotic smoothness. We will support this conjecture by showing that a 4-manifold admitting a Ricci-flat metric (in standard smoothness structure) changes to a 4-manifold with non-Ricci-flat metric in all exotic smoothness structures (see the discussion in subsection 4.1). Thus *if one starts with a vacuum solution of Einstein's equation then exotic smoothness modifies this solution to a non-vacuum solution, i.e. the sources are determined by the exotic smoothness structure*. In the following we will study this relation more carefully by showing how exotic smoothness is able to generate the known action terms.

1.1 Outline of the paper

At first we have to note that there are many examples of exotic, *non-compact* 4-manifolds which are hard to describe at the present state of knowledge. Thus we will restrict ourselves to the class of *compact* 4-manifolds where one has powerful invariants like Seiberg-Witten invariants [1, 59] or Donaldson polynomials [19, 20] to distinguish between different smoothness structures. A recent result [60] can be used to produce a non-compact exotic 4-manifold (distinguished by the Seiberg-Witten invariants) which embeds in the compact 4-manifold considered. So, if we have a compact 4-manifold M then the non-compact 4-manifold $M \setminus B^4$ (with the open 4-ball $B^4 = D^4 \setminus \partial D^4$) embeds obviously

in M . Especially if M is exotic then $M \setminus B^4$ is exotic too. Additionally the non-compact 4-manifold $M \setminus B^4$ admits a Lorentz metric.

Secondly, we will not discuss the definition of the path integral and all problems connected with renormalization, definiteness etc.

Third, we use the knot surgery of Fintushel and Stern [24] to construct the exotic smoothness structure. This approach assumes a special class of 4-manifolds (“complicated-enough”) which contains the class of elliptic surfaces among them the important K3 surface.

This leading us to the assumptions of the 4-manifold representing the spacetime of the model:

1. The compact and simple-connected 4-manifold under consideration is an elliptic surface $E(n)$, especially $E(2)$ which is the *K3 surface*.
2. The non-compact 4-manifold $E(n) \setminus B^4$ admits a Lorentz metric and represents the spacetime of the model.
3. The exotic smoothness structures of the 4-manifold $E(n)$ are constructed by Fintushel-Stern knot surgery. It follows that this surgery is also producing an exotic smoothness structure on $E(n) \setminus B^4$.

In section 4 we will describe the physical model and motivate it. As shown by LeBrun [42, 43] using Seiberg-Witten theory, a 4-manifold as above admitting an Einstein metric in the standard smooth structure fails to admit an Einstein metric in some exotic smooth structure. Therefore a spacetime without matter can be changed to a spacetime with source terms by changing the smoothness structure. Together with the confirmed Brans conjecture, that exotic smoothness is a source of gravity, we have a good motivation for our main hypothesis: *matter as represented by fields is emerged from exotic smoothness*. With this motivation in mind, we start with a 4-manifold $M = E(2)$ in standard structure and apply the Fintushel-Stern knot surgery to get the exotic 4-manifold M_K and consider the Einstein-Hilbert action on M_K agreeing with the Einstein-Hilbert action on M outside of some compact submanifold. Then we will discuss in section 5 a decomposition of the 4-manifold M_K which induces by using the diffeomorphism invariance a decomposition of the action. In this decomposition the knotted torus of the knot surgery corresponds to the Dirac action via its Weierstrass (or spinor) representation, described in section 6. For more complicated knots like satellite knots or sums of knots we obtain a splitting of the knot complement into simpler pieces clued together along connecting tubes. In section 7 we show that this connecting tubes represent the gauge fields and in section 8 we will speculate about the derivation of the gauge group using the classification of the connecting tubes. A short discussion of the results in the last section completes the paper.

2 Lorentz metric and global hyperbolicity

Before we start the investigation of the proposed model, we will discuss some more general physical implications. Firstly we consider the existence of a

Lorentz metric, i.e. a 4-manifold M (the spacetime) admits a Lorentz metric if (and only if) there is a non-vanishing vector field. In case of a compact 4-manifold M we can use the Poincare-Hopf theorem to state: a compact 4-manifold admits a Lorentz metric if the Euler characteristic vanishes $\chi(M) = 0$. But in a compact 4-manifold there are closed time-like curves (CTC) contradicting the causality or more exactly: the chronology violating set of a compact 4-manifold is non-empty (Proposition 6.4.2 in [38]). Non-compact 4-manifolds M admit always a Lorentz metric and a special class of these 4-manifolds have an empty chronology violating set. If \mathcal{S} is an acausal hypersurface in M (i.e., a topological hypersurface of M such that no pair of points of M can be connected by means of a causal curve), then $D^+(\mathcal{S})$ is the future Cauchy development (or domain of dependence) of \mathcal{S} , i.e. the set of all points p of M such that any past-inextendible causal curve through p intersects \mathcal{S} . Similarly $D^-(\mathcal{S})$ is the past Cauchy development of \mathcal{S} . If there are no closed causal curves, then \mathcal{S} is a Cauchy surface if $D^+(\mathcal{S}) \cup \mathcal{S} \cup D^-(\mathcal{S}) = M$. As shown in [9], the existence of a Cauchy surface implies that M is diffeomorphic to $\mathcal{S} \times \mathbb{R}$.

This strong result is also connected with the concept of global hyperbolicity. A spacetime manifold M without boundary is said to be *globally hyperbolic* if the following two conditions hold:

1. *Absence of naked singularities*: For every pair of points p and q in M , the space of all points that can be both reached from p along a past-oriented causal curve and reached from q along a future-oriented causal curve is compact.
2. *Chronology*: No closed causal curves exist (or "Causality" holds on M).

Usually condition 2 above is replaced by the more technical condition "Strong causality holds on M " but as shown in [8] instead of "strong causality" one can write simply the condition "causality" (and strong causality will hold under causality plus condition 1 above).

Then together with the diffeomorphism between M and $\mathcal{S} \times \mathbb{R}$ we can conclude that all (non-compact) 4-manifolds $\mathcal{S} \times \mathbb{R}$ are the only 4-manifolds which admit a globally hyperbolic Lorentz metric [9]. The existence of a Cauchy surface \mathcal{S} implies global hyperbolicity of the spacetime and its unique representation by $\mathcal{S} \times \mathbb{R}$ (up to diffeomorphism). But as shown in [7], also the metric is determined (up to isometry) by global hyperbolicity.

Theorem 1 *If a spacetime (M, g) is globally hyperbolic, then it is isometric to $(\mathbb{R} \times \mathcal{S}, -f \cdot d\tau^2 + g_\tau)$ with a smooth positive function $f : \mathbb{R} \rightarrow \mathbb{R}$ and a smooth family of Riemannian metrics g_τ on \mathcal{S} varying with τ . Moreover, each $\{t\} \times \mathcal{S}$ is a Cauchy slice.*

Furthermore in [10] it was shown:

- If a compact spacelike submanifold with boundary of a globally hyperbolic spacetime is acausal then it can be extended to a full Cauchy spacelike hypersurface \mathcal{S} of M , and

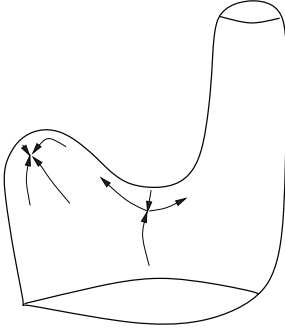


Fig. 1 Two naked singularities.

- for any Cauchy spacelike hypersurface \mathcal{S} there exists a function as in Th. 1 such that \mathcal{S} is one of the levels $\tau = \text{constant}$.

But what are the implications of global hyperbolicity in the exotic case? At first, the existence of a Lorentz metric is a purely topological condition which will be fulfilled by all non-compact 4-manifolds independent of the smoothness structure. But by considering global hyperbolicity the picture changes. An exotic spacetime $M = (\mathcal{S} \times \mathbb{R})_{\text{exotic}}$ homeomorphic to $\mathcal{S} \times \mathbb{R}$ is *not diffeomorphic* to $\mathcal{S} \times \mathbb{R}$. The Cauchy surface \mathcal{S} is a 3-manifold with an unique smoothness structure (up to diffeomorphisms) – the standard structure. So, the smooth product $\mathcal{S} \times \mathbb{R}$ must admit the standard smoothness structure. But the diffeomorphism [9] between M and $\mathcal{S} \times \mathbb{R}$ is necessary for global hyperbolicity. Therefore an *exotic* $(\mathcal{S} \times \mathbb{R})_{\text{exotic}}$ is *never globally hyperbolic but admits a Lorentz metric*. Generally we have an exotic $(\mathcal{S} \times \mathbb{R})_{\text{exotic}}$ with a Lorentz metric such that the projection $(\mathcal{S} \times \mathbb{R})_{\text{exotic}} \rightarrow \mathbb{R}$ is a time-function (that is, a continuous function which is strictly increasing on future directed causal curves). But then the exotic $(\mathcal{S} \times \mathbb{R})_{\text{exotic}}$ has no closed causal curves and must contain naked singularities¹.

With this result in mind, one should ask for the physical interpretation of naked singularities. To visualize the problem, we consider the following toy model: a non-trivial surface (see Fig. 1) connecting two circles which can be deformed to the usual cylinder. This example can be described by the concept of a cobordism. A cobordism (W, M_1, M_2) between two n -manifolds M_1, M_2 is a $(n+1)$ -manifold W with $\partial W = M_1 \sqcup M_2$ (ignoring the orientation). Then there exists a smooth function $f : W \rightarrow [0, 1]$ with isolated critical points (vanishing first derivative) such that $f^{-1}(0) = M_1$, $f^{-1}(1) = M_2$. By general position arguments, one can assume that all critical points of f occur in the

¹ Any non-compact manifold M admits stably causal metrics (that is, those with a time function). So, if M is not diffeomorphic to some product $\mathcal{S} \times \mathbb{R}$, all these (causally well behaved) metrics must contain naked singularities. We thank M. Sánchez for the explanation of this result.

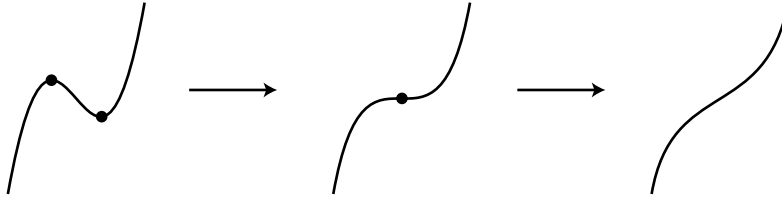


Fig. 2 Killing a 0- and a 1-handle.

interior of W . In this setting f is called a Morse function on a cobordism. For every critical point of f (vanishing first derivative) one adds a so-called k -handle $D^k \times D^{n-k}$. In our example in Fig. 1, we add a 2-handle $D^2 \times D^0$ (the maximum) and a 1-handle $D^1 \times D^1$ (the saddle). But obviously this cobordism is diffeomorphic to the trivial cobordism $S^1 \times [0, 1]$ (the two boundary components are diffeomorphic to each other). Therefore the 2-/1-handle pair is "killed" in this case. The 2-handle and the 1-handle differ in one direction whereas the Morse function has a maximum for the 2-handle and a minimum for the 1-handle. The left graph of Fig. 2 visualizes this fact. Furthermore the sequence of graphs from the left to right represents the process to "kill" the handle pair.

In our example of an exotic $(\mathcal{S} \times \mathbb{R})_{\text{exotic}}$, we consider a (non-compact) cobordism between homeomorphic boundary components (the two Cauchy surfaces at infinity, i.e. " $\mathcal{S} \times \{-\infty\}$ " and " $\mathcal{S} \times \{+\infty\}$ "). By a result of [45], a cobordism of this kind (a so-called h-cobordism) contains only handles of complement dimension in the interior. But these handles can be killed: the details of the construction can be found in [45]. Here we will give only some general remarks. Any 0-/1-handle pair as well any $n/(n+1)$ -handle pair (remember the h-cobordism is $n+1$ -dimensional) can be killed by a general procedure. The killing of a $k/(k+1)$ -handle pair depends on a special procedure, the Whitney trick. For 4- and 5-dimensional h-cobordisms (between 3- and 4-manifolds, respectively) we cannot use the Whitney trick. This failure lies at the heart of the problem to classify 3- and 4-manifolds (see [37]).

In case of the spacetime we are interested in a 4-dimensional h-cobordism between 3-manifolds. Here we can kill the 0-/1- and the 3-/4-handle pair of the h-cobordism and we are left with pairs of 2-handles. If the Whitney trick works in this case, we can also kill these pairs of handles. But it is known that the Whitney trick only works topologically [29]. The exotic manifolds $(\mathcal{S} \times \mathbb{R})_{\text{exotic}}$ (as non-compact examples) are counterexamples thus the pairs of 2-handles never cancel each other. The critical point of the Morse function with index 2 (the Morse function has a minimum in two directions (saddle point)) corresponds to this 2-handle. Each pair of 2-handles is connected to each other, i.e. the directions representing the minimum of a 2-handle are connected with the directions representing the maximum of the other 2-handle. Each 2-handle represents the neighborhood of a naked singularity. These singularities appear only pairwise. The Morse vector field (gradient of the Morse function) vanishes but we get a saddle having negative curvature (*no curvature*

singularity or *quasiregular singularity*, also called *locally extendible singularity* [22]). There is growing evidence for the appearance of this singularity without violating causality (causal continuity, see [21]). Thus it is very probable that the property of exotic smoothness requiring the appearance of naked singularities in form of non-canceling 2-handle pairs can be interpreted in a consistent physically way: In the following sections we will merge exotic smoothness with fermion and bosons obtaining the consequence that exotic smoothness implies naked singularities as well as physically particles. We do not think this is an accident – we conjecture that the naked singularities can be seen as the geometrically consequence of the physically particles.

In this paper we propose a model where an open 4-ball B^4 ($B^4 = D^4 \setminus \partial D^4$) is removed from the compact 4-manifold M to obtain the open 4-manifold $M \setminus B^4$. This 4-manifold admits a Lorentz metric but contains also naked singularities. In the following we discuss the case of a compact 4-manifolds admitting a Ricci-flat metric, the K3 surface. The choice has a conceptional background: If we interpret matter as geometrical objects then there is no need for a non-geometrical energy-momentum tensor. Therefore we have to consider the source-free Einstein equation implying Ricci-flatness. But there are only two compact 4-manifolds with Ricci-flat metric, the 4-torus and the K3 surface. This result is only true for a Riemannian metric but we can introduce a Lorentz metric for the K3 surface and the 4-torus by removing an open 4-ball (see above). The 4-torus is a flat manifold, i.e. the curvature vanishes. From the physical point of view, it is useless to choose this manifold. Therefore we will concentrate on the K3 surface (minus an open 4-ball) which is a so-called elliptic surface.

3 Preliminaries: Elliptic surfaces and exotic smoothness

In this section we will give some information about elliptic surfaces and its construction. First we will discuss smoothness in general and give a short overview of the construction of elliptic surfaces and a special class of elliptic surfaces denoted by $E(n)$ in the literature. Then we present the construction of exotic $E(n)$ by using the knot surgery of Fintushel and Stern.

3.1 Smoothness on manifolds

From the mathematical point of view, the spacetime is a smooth 4-manifold endowed with a (smooth) metric as basic variable for general relativity. In the previous section we discuss the existence question for Lorentz structure and causality problems (see Hawking and Ellis [38]) giving further restrictions on the 4-manifold: causality implies non-compactness, Lorentz structure needs a non-vanishing normal vector field. The appropriate notation is the global hyperbolic 4-manifold M having a Cauchy surface \mathcal{S} so that $M = \mathcal{S} \times \mathbb{R}$.

All these restrictions on the representation of spacetime by the manifold concept are clearly motivated by physical questions. Among the properties

there is one distinguished element: the smoothness. Usually one starts with a topological 4-manifold M and introduces structures on them. Then one has the following ladder of possible structures:

$$\begin{aligned} \text{Topology} &\rightarrow \text{piecewise-linear(PL)} \rightarrow \text{Smoothness} \rightarrow \\ &\rightarrow \text{bundles, Lorentz, Spin etc.} \rightarrow \text{metric, geometry, ...} \end{aligned}$$

We do not want to discuss the first transition, i.e. the existence of a triangulation on a topological manifold. But we remark that the existence of a PL structure implies uniquely a smoothness structure in all dimensions smaller than 7 [40]. Here we have to consider the following steps to define a spacetime:

1. Fix a topology for the spacetime M .
2. Fix a smoothness structure, i.e. a maximal differentiable atlas \mathcal{A} .
3. Fix a smooth metric or get one by solving the Einstein equation.

The choice of a topology never fixes the spacetime uniquely, i.e. there are two spacetimes with the same topology which are not diffeomorphic. The following basic facts should the reader keep in mind for any n -dimensional manifold M^n :

1. The maximal differentiable atlas \mathcal{A} of M^n is the smoothness structure.
2. To determine a smoothness structure it suffices to give a single maximal differentiable atlas. Thus \mathbb{R}^n has a unique smoothness structure containing the identity map of \mathbb{R}^n (standard smoothness structure, $(\mathbb{R}^n, id_{\mathbb{R}^n})$ is the atlas).
3. Every manifold M^n can be embedded in \mathbb{R}^N with $N > 2n$. An embedding $M^n \hookrightarrow \mathbb{R}^N$ induces the *standard smoothness structure* on M . The \mathbb{R}^N has an unique smoothness structure – the standard structure. All other possible smoothness structures non-diffeomorphic to the standard smoothness structure are called *exotic* smoothness structures.
4. The existence of a smoothness structure is *necessary* to introduce Riemannian or Lorentzian structures on M , but the smoothness structure do not further restrict the Lorentz structure.

We want to close this subsection with a general remark: the number of non-diffeomorphic smoothness structures is finite for all dimensions $n \neq 4$ [40]. In dimension four there are many examples of compact 4-manifolds with infinite finite and many examples of non-compact 4-manifolds with uncountable infinite many non-diffeomorphic smoothness structures.

3.2 Elliptic surfaces

A *complex surfaces* S is a 2-dimensional complex manifold which is compact and connected. A special complex surface is the *elliptic surface*, i.e. a complex surface S together with a map $\pi : S \rightarrow C$ (C complex curve, i.e. Riemannian surface), so that for nearly every point $p \in S$ the reversed map $F = \pi^{-1}(p)$

is an elliptic curve, i.e. a torus². Now we will construct the special class of elliptic surfaces $E(n)$.

The first step is the construction of $E(1)$ by the unfolding of singularities for two cubic polynomials intersecting each other. The resulting manifold $E(1)$ is the manifold $CP^2 \# 9\overline{CP}^2$ but equipped with an elliptic fibration. Then we use the method of fiber sum to produce the surfaces $E(n)$ for every number $n \in \mathbb{N}$. For that purpose we cut out a neighborhood $N(F)$ of one fiber $\pi^{-1}(p)$ of $E(1)$. Now we sew together two copies of $E(1) \setminus N(F)$ along the boundary of $N(F)$ to get $E(2)$, i.e. we define the fiber sum $E(2) = E(1) \#_f E(1)$. Especially we note that $E(2)$ is also known as *K3-surface* widely used in physics. In general we get the recursive definition $E(n) = E(n-1) \#_f E(1)$. The details of the construction can be found in the paper [36].

3.3 Knot surgery and exotic elliptic surface

The main technique to construct an exotic elliptic surface was introduced by Fintushel and Stern [24], called *knot surgery*. In short, given a simply-connected, compact 4-manifold M with an embedded torus T^2 (having special properties, see below), cut out $M \setminus N(T^2)$ a neighborhood $N(T^2) = D^2 \times T^2$ of the torus and glue in $S^1 \times (S^3 \setminus N(K))$. The 3-manifold $S^3 \setminus N(K)$ is called the *knot complement* of K (see appendix A)³. This surgery construction depends only on the gluing operation of the knot K , i.e. an embedding of the circle S^1 into \mathbb{R}^3 or S^3 , and one obtains the new 4-manifold

$$M_K = (M \setminus N(T)) \cup_{T^3} (S^1 \times (S^3 \setminus N(K)))$$

from a given 4-manifold M by gluing $M \setminus N(T^2)$ and $S^1 \times (S^3 \setminus N(K))$ along the common boundary, the 3-torus T^3 . The remarkable result of Fintushel and Stern [24] is, that a gluing with a non-trivial knot K changes M non-diffeomorphic to M_K . We remark that the construction can be easily generalized to links, i.e. the embedding of the disjoint union of circles $S^1 \sqcup \dots \sqcup S^1$ into \mathbb{R}^3 or S^3 . The reader not interested in the details of the construction can now jump to the next section.

The precise definition can be given in the following way for elliptic surface: Let $\pi: S \rightarrow C$ be an elliptic surface and $\pi^{-1}(t) = F$ a smooth fiber ($t \in C$). As usual, $N(F)$ denotes a neighborhood of the regular fiber F in S (which is diffeomorphic to $D^2 \times T^2$). Deleting $N(F)$ from S to get a manifold $S \setminus N(F)$ with boundary $\partial(S \setminus N(F)) = \partial(D^2 \times F) = S^1 \times F = T^3$, the 3-torus. Then we take the 4-manifold $S^1 \times (S^3 \setminus N(K))$, K a knot, with boundary $\partial(S^1 \times (S^3 \setminus N(K))) = T^3$ and regluing it along the common boundary T^3 . The resulting 4-manifold

$$S_K = (S \setminus N(F)) \cup_{T^3} (S^1 \times (S^3 \setminus N(K)))$$

² We denote this map π as elliptic fibration. Thus every complex surface which is equipped with a elliptic fibration is an elliptic surface.

³ Alternatively, the knot complement is the 3-manifold obtained by performing 0-framed surgery on the knot.

is obtained from S by knot surgery using the knot K . The regular fiber F in the elliptic surface S has two properties which are essential for the whole construction:

1. In a larger neighborhood $N_c(F)$ of the regular fiber F there is a cups fiber c , i.e. an embedded 2-sphere of self-intersection 0 with a single, non-locally flat point whose neighborhood is the cone over the right-hand trefoil knot (so-called c -embedded torus).
2. The complement $S \setminus F$ of the regular fiber is simple-connected $\pi_1(S \setminus F) = 1$.

Then as shown in [24], S_K is not diffeomorphic to S . The whole procedure can be generalized to any 4-manifold allowing an embedding of a torus of self-intersection 0 in a neighborhood of a cusp.

Before we proceed with the physical interpretation, we will discuss the question when two exotic S_K and $S_{K'}$ for two knots K, K' are diffeomorphic to each other. Currently there are two invariants to distinguish non-diffeomorphic smoothness structures: Donaldson polynomials and Seiberg-Witten invariants. Fintushel and Stern [24] calculated the Seiberg-Witten invariants for S_K and $S_{K'}$ to show that S_K differs from $S_{K'}$ if the Alexander polynomials (a knot invariant, see [52]) of the two knots differ. Unfortunately this invariant is not a complete classifying invariant for knots. Thus we cannot say anything about S_K and $S_{K'}$ for two knots with the same Alexander polynomial. But Fintushel and Stern [26, 27] constructed counterexamples of two knots K, K' with the same Alexander polynomial but with different S_K and $S_{K'}$. Furthermore Akbulut [2] showed that the knot K and its mirror \bar{K} induce diffeomorphic 4-manifold $S_K = S_{\bar{K}}$.

In a recent paper [60], the non-compact case was also discussed. Consider the non-compact elliptic surface $S_K \setminus B^4$ which embeds in S_K . By Theorem 1.1 in [60]: if the Seiberg-Witten invariant of S_K do not vanish (which is true for a knot K with non-trivial Alexander polynomial) then $S_K \setminus B^4$ admits infinitely many distinct exotic smooth structures.

4 Matter from exotic spacetime

4.1 Motivation

In this section we will motivate our hypothesis: matter emerges from exotic spacetime. We start with the Brans conjecture: exotic smoothness induces additional sources of gravity. One of the authors proved this conjecture for compact manifolds [5] generating source terms in Einstein field equation. Then Sladkowski [53] showed the conjecture for the exotic \mathbb{R}^4 , i.e. the exoticness implies non-flat solutions of the Einstein field equation. Now we will discuss properties of matter and its realization by exotic smoothness:

1. Locality: The smoothness structure is by definition a global structure – the maximal differentiable atlas. But in knot surgery, a local modification changes the smoothness structure.

2. Infinite trajectories: As discussed above, we consider a non-compact 4-manifold as spacetime. Therefore, we have trajectories which can be extended to infinity.
3. Stability: Every knot surgery enforces a modification of the smoothness structure to a non-diffeomorphic one in relation to the smoothness structure at the starting point. Therefore it seems that exotic smoothness is a relative phenomenon which can be gauged away – but this is wrong. The necessary transformation for changing the smoothness is a non-diffeomorphism, i.e. one changes the whole physical system by using this transformation. Secondly, we will show in this paper that every exotic smoothness structure generates matter. Then the standard smoothness structure represents the spacetime without matter.

In subsection 3.3 we discuss the smoothness structures for a compact, simple-connected, smooth 4-manifold by using the Seiberg-Witten invariant [59, 1]. It is known that an exotic smoothness structure implies a non-trivial Seiberg-Witten invariant, i.e. this invariant vanishes for the standard structure. While the invariant is not complete (two different exotic smoothness structures can have the same invariant) it is able to distinguish between the standard smoothness structure and any exotic one.

In this paper we will consider the K3 surface $E(2)$ and choose its non-compact version $E(2) \setminus B^4$ as spacetime model. The choice of the K3 surface is not arbitrary: the K3 surface is the only compact, simple-connected, closed 4-manifold with Ricci-flat metric [11] (see also the section 2). This Ricci-flat metric is a Riemannian metric but this is no problem because the K3 surface itself do not admit a Lorentz metric at all. Hence one has to choose as a model of the spacetime the non-compact version of the K3 surface $E(2) \setminus B^4$ admitting a Lorentz metric. In the usual view, matter is represented by a source-term of the Einstein equation. In the geometrical model proposed here it should be an expression of exotic smoothness only without the help of conventional (non-geometrical) source-terms. Thus to investigate the effect of exoticness only we have to start with a plain spacetime in standard smoothness structure and with vacuum metric, i.e. an Einstein equation without any conventional matter-terms. Then we can evaluate the effect of the exoticness by switching this vacuum spacetime to an exotic smoothness structure. For this propose our starting spacetime has to admit a vacuum solution, i.e. has to be Ricci-flat. As mention above, the K3 surface is the only compact, simple-connected, closed 4-manifold with Ricci-flat metric. This motivates the choice.

4.2 The model

In this section we will discuss the additional contribution to the Einstein-Hilbert action functional coming from exotic smoothness generated by knot surgery. Our model starts with an elliptic surface, the K3 surface $E(2)$ motivated in the previous subsection. We choose the non-compact 4-manifold $M = E(2) \setminus B^4$ as spacetime, admitting a Ricci-flat metric g . The work in

[60] relates the smoothness properties of $E(2)$ to M . Especially a knot surgery on $E(2)$ produces an exotic smoothness structure (if the knot has non-trivial Alexander polynomial) as well on the spacetime M . Here we consider the Einstein-Hilbert action

$$S_{EH}(g) = \int_M R\sqrt{g} d^4x \quad (1)$$

with the Ricci-flat metric g as solution of the vacuum field equations and study the effect of switching the smooth structure by a knot surgery. This procedure touches only a submanifold $N(T^2) \subset M$ and thus we consider a decomposition of the 4-manifold

$$M = (M \setminus N(T^2)) \cup_{T^3} N(T^2)$$

with $N(T^2) = D^2 \times T^2$ leading to a sum in the action

$$S_{EH}(M) = \int_{M \setminus N(T^2)} R\sqrt{g} d^4x + \int_{N(T^2)} R\sqrt{g} d^4x \quad .$$

Because of diffeomorphism invariance of the Einstein-Hilbert action, this decomposition do not depend on the concrete realization with respect to any coordinate system. Now we switch to a new smoothness structure on M by using a knot $K : S^1 \rightarrow S^3$

$$M_K = (M \setminus N(T^2)) \cup_{T^3} (S^1 \times (S^3 \setminus N(K)))$$

i.e. we exchange $N(T^2)$ by $S^1 \times (S^3 \setminus N(K))$ and call the resulting exotic 4-manifold M_K . The 4-manifold $M \setminus N(T^2)$ with boundary a 3-torus T^3 appears in both 4-manifolds M and M_K . Thus we can fix its action

$$S_{EH}(M \setminus N(T^2)) = \int_{M \setminus N(T^2)} R\sqrt{g} d^4x$$

by using a fixed metric g in the interior $\text{int}(M \setminus N(T^2))$. Furthermore a possible boundary term can be ignored because the 3-torus $T^3 = \partial(M \setminus N(T^2))$ is a flat, compact 3-manifold. The decomposition of M_K gives for the action

$$S_{EH}(M_K) = S_{EH}(M \setminus N(T^2)) + \int_{S^1 \times (S^3 \setminus N(K))} R_K \sqrt{g_K} d^4x \quad (2)$$

with a metric g_K and scalar curvature R_K of the 4-manifold $S^1 \times (S^3 \setminus N(K))$. To determine the first term in (2) we consider the integral

$$S_{EH}(N(T^2)) = \int_{N(T^2)=D^2 \times T^2} R\sqrt{g} d^4x$$

over $N(T^2) = D^2 \times T^2$ w.r.t. a suitable product metric. The torus T^2 is a flat manifold and the disk D^2 can be chosen to embed flatly in $N(T^2)$. Thus, this integral vanishes $S_{EH}(N(T^2)) = 0$ and

$$S_{EH}(M \setminus N(T^2)) = S_{EH}(M) \quad .$$

Using this and (2), we obtain the relation

$$S_{EH}(M_K) = S_{EH}(M) + \int_{S^1 \times (S^3 \setminus N(K))} R_K \sqrt{g_K} d^4x \quad (3)$$

between the Einstein-Hilbert action on M and M_K showing that the *contribution of the exotic smoothness structure to the Einstein-Hilbert action* is given by the action integral over $S^1 \times (S^3 \setminus N(K))$. We evaluate the exotic action

$$S_{exotic} \doteq \int_{S^1 \times (S^3 \setminus N(K))} R_K \sqrt{g_K} d^4x \quad (4)$$

by using a product metric g_K

$$ds^2 = d\theta^2 + h_{ik} dx^i dx^k \quad (5)$$

with periodic coordinate θ on S^1 and metric h_{ik} on the knot complement $S^3 \setminus N(K)$. By the ADM formalism with the lapse N and shift function N^i one gets a relation between the 4-dimensional R_K and the 3-dimensional scalar curvature $R_{(3)}$ (see [46] (21.86) p. 520)

$$\sqrt{g_K} R_K d^4x = N \sqrt{h} (R_{(3)} + \|n\|^2 ((tr \mathbf{K})^2 - tr \mathbf{K}^2)) d\theta d^3x \quad (6)$$

with the normal vector n and the extrinsic curvature \mathbf{K} . The 4-manifold $S^1 \times (S^3 \setminus N(K))$ has a product structure. Because of the diffeomorphism invariance of the action, we can choose a special coordinate system but get always the same result. The product metric in $S^1 \times (S^3 \setminus N(K))$ allows an embedding $S^3 \setminus N(K) \hookrightarrow S^1 \times (S^3 \setminus N(K))$ in such a manner that the extrinsic curvature do not depend on θ and it can be chosen to be constant $\mathbf{K} = const.$ One obtains

$$S_{exotic} = L_{S^1} \cdot \left(\int_{(S^3 \setminus N(K))} R_{(3)} \sqrt{h} N d^3x \right) \quad (7)$$

the 3-dimensional Einstein-Hilbert action times the length $L_{S^1} = \int_{S^1} d\theta$ of the circle S^1 . Applying the identity

$$S^3 = (S^3 \setminus N(K)) \cup N(K)$$

to the integrals one obtains

$$\int_{(S^3 \setminus N(K))} R_{(3)} \sqrt{h} N d^3x + \int_{N(K)} R_{(3)} \sqrt{h} N d^3x = \int_{S^3} R_{(3)} \sqrt{h} N d^3x = \lambda \cdot vol(S^3)$$

with a constant curvature $\lambda > 0$ of the 3-sphere and so

$$\int_{(S^3 \setminus N(K))} R_{(3)} \sqrt{h} N d^3x = \lambda \cdot \text{vol}(S^3) - \int_{N(K)} R_{(3)} \sqrt{h} N d^3x.$$

The integral over $N(K) = K \times D^2$ is completely determined by the boundary (the disk is flatly embedded). It can be calculated as a term over the boundary $\partial N(K) = K \times S^1$, a knotted torus, i.e. we obtain

$$\begin{aligned} S_{EH}(N(K)) &= \int_{N(K)} R_{(3)} \sqrt{h} N d^3x = \int_{\partial(N(K))} X \sqrt{h} d^2x \\ &= S_{EH}(\partial(N(K))) \end{aligned}$$

where X is a 2-dimensional expression for the boundary term of the Einstein-Hilbert action. We will use the same symbol for the 2-dimensional metric h and its restriction to the boundary submanifold. Now we are looking for the action at the boundary. As shown by York [61], the fixing of the conformal class of the spatial metric in the ADM formalism leads to a boundary term which can be also found in the work of Hawking and Gibbons [31]. Also Ashtekar et.al. [3,4] discussed the boundary term in the Palatini formalism. The main reason for the introduction of the boundary term came from the Hamiltonian formulation of Einsteins theory. It has been known since the 1960's (see [46] section 21.4-21.8) that in the Hamiltonian quantization of gravity it is essential to include boundary terms in the action, as this allows to define consistently the momentum conjugate to the metric. This makes it necessary to modify the Einstein-Hilbert action by adding to it a surface integral term so that the variation of the action becomes well defined and yields the Einstein field equations. All these discussions enforce us to choose the following action term at the boundary $\partial(N(K))$

$$S_{EH}(\partial(N(K))) = \int_{\partial(N(K))} H_{\partial} \sqrt{h} d^2x$$

with H_{∂} as *mean curvature* of $\partial(N(K))$, i.e. the trace of the second fundamental form. Of course the mean curvature do not depend on the coordinate θ of the circle S^1 above. Therefore we obtain for the exotic action (4)

$$S_{exotic} = \lambda \cdot \text{vol}(S^1 \times S^3) - \int_{S^1 \times \partial(N(K))} H_{\partial} \sqrt{h} d\theta d^2x \quad (8)$$

with the constant λ representing the curvature of $S^1 \times S^3$. The manifold $S^1 \times \partial N(K)$ is a knotted 3-torus $T^3(K) = K \times S^1 \times S^1$.

Finally we found, that the contribution of the exotic smooth structure to the usual Einstein-Hilbert action $S_{EM}(M)$ is given by an additive term – the exotic action S_{exotic} which is a sum of two Einstein-Hilbert terms, one over

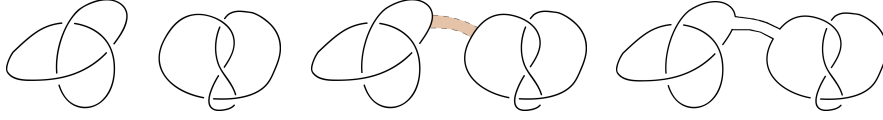


Fig. 3 Connected sum of knots (left: start, middle: connecting rectangle, right: join of knots)

$S^1 \times S^3$ with constant curvature and another one over a knotted 3-torus with mean curvature H_{∂}

$$S_{EH}(M_K) = S_{EH}(M) + S_{exotic} \quad (9)$$

$$= S_{EH}(M) + \lambda \cdot \text{vol}(S^1 \times S^3) - \int_{S^1 \times \partial(N(K))} H_{\partial} \sqrt{h} d\theta d^2x \quad (10)$$

and in usual units (L_P Planck length) one has

$$\frac{1}{\hbar} S_{EH}(M_K) = \frac{1}{\hbar} S_{EH}(M) + \frac{1}{L_P^2} \cdot \left(\lambda \cdot \text{vol}(S^1 \times S^3) - \int_{S^1 \times \partial(N(K))} H_{\partial} \sqrt{h} d\theta d^2x \right). \quad (11)$$

In the next section we will show that these exotic action terms have a common geometrical meaning.

5 Geometrical Matter

Lets start with the boundary term

$$\int_{S^1 \times \partial(N(K))} H_{\partial(N(K))} \sqrt{h} d\theta d^2x$$

in (11) over the knotted 3-torus $T^3(K) = S^1 \times \partial N(K)$. The value of this integral depends strongly on the knot K and we have to say some words about the complexity of knots. For that purpose we define the connected sum of two oriented knots:

- Consider a planar projection of each knot and suppose these projections are disjoint. (see left figure of Fig. 3)
- Find a rectangle in the plane where one pair of sides are arcs along each knot but is otherwise disjoint from the knots and the arcs of the knots on the sides of the rectangle are oriented around the boundary of the rectangle in the same direction. (see middle figure of Fig. 3)
- Now join the two knots together by deleting these arcs from the knots and adding the arcs that form the other pair of sides of the rectangle. (see right figure of Fig. 3)

The resulting connected sum knot inherits an orientation consistent with the orientations of the two original knots. With respect to this operation there is now a prime factorization: a non-trivial knot is decomposable if one can represent it by a sum of other non-trivial knots. A prime knot is a non-trivial knot which cannot be written as the knot sum of two non-trivial knots. A theorem due to Schubert [52] states that every knot can be uniquely expressed as a connected sum of prime knots. Therefore we have to concentrate on the prime knots. But one has to respect how this operation affects the knot complement in the LHS of (8). For that purpose we consider the integral

$$\int_{(S^3 \setminus N(K))} R_{(3)} \sqrt{h} N d^3 x$$

over the knot complement $S^3 \setminus N(K)$. The knot complements are 3-manifolds with a boundary which is a torus. Like for knots, there is also a prime decomposition for 3-manifolds. A 3-manifold P is prime if it cannot be presented as a connected sum in a non-trivial way, i.e. not like $P = P \# S^3$. Then [44], the prime decomposition theorem for 3-manifolds states that every compact, orientable 3-manifold is the connected sum of a unique (up to homeomorphism) collection of prime 3-manifolds. Knot complements are always prime manifolds. But it is also possible to split them according to the splitting of a general knot into a sum of prime knots. Budney [17] considers the various operations of knots and its corresponding operation on the knot complement. Let $C(K) = S^3 \setminus N(K)$ be the knot complement for the knot K and assume for K a sum

$$K = K_1 \# K_2$$

of prime knots K_1, K_2 . Then the knot complements admits a splitting [17]

$$C(K) = C(K_1) \cup_{T^2} T(K_1, K_2) \cup_{T^2} C(K_2).$$

We call $T(K_1, K_2)$ the *connecting tube* between the knot complements $C(K_1)$ and $C(K_2)$. This 3-manifold $T(K_1, K_2)$ is a so-called graph manifold (or Seifert fibered). In our case, it can be described by the link complement $T(K_1, K_2) = S^3 \setminus N(H^2)$ with the so-called key chain link H^2 (a generalization of a Hopf link). The connecting tube $T(K_1, K_2)$ has a boundary consisting of three disjoint tori $\partial T(K_1, K_2) = T_1^2 \sqcup T_2^2 \sqcup T_3^2$ (we ignore the orientation) where one of these tori T_3^2 is the boundary $\partial C(K) = T_3^2$ of $C(K)$. If we ignore this boundary (by closing it with a solid torus $T(K_1, K_2) \cup_{T_3^2} (D^2 \times S^1)$) then we have a trivial torus bundle $T^2 \times [0, 1]$ between T_1^2 and T_2^2 . The most general operation on knots is the splicing $J \bowtie K$ (see for the details [17]) producing examples of a splitting

$$C(J \bowtie K) = C(K) \cup_{T^2} T(K, J) \cup_{T^2} C(J)$$

with non-trivial torus bundle $T(K, J)$ (by closing one boundary) between $C(K)$ and $C(J)$.

The reason why we consider the details of the splitting of knot complements is that knot complements as prime 3-manifolds are the main objects of Thurston's geometrization conjecture [55] (now proven by Perelman [48, 50, 49]). This knot complement admits a geometric structure⁴ in the interior $C(K) \setminus \partial C(K)$, i.e. a homogeneous metric of constant scalar curvature (a Bianchi model in physical notation). The knots divide into two classes:

1. hyperbolic knots: the knot complement is a hyperbolic 3-manifold and
2. non-hyperbolic knots: the knot complements admits one of the 7 other geometric structures.

We would like conjecture that this classification of knots fits well with the classification of matter into fermions and bosons. Let K be a hyperbolic knot with its hyperbolic complement $C(K)$. Hyperbolic 3-manifolds are subject to a strong restriction called *Mostow rigidity* [47]. It states that any diffeomorphism (including a conformal map) of a hyperbolic 3-manifold is an isometry. Thus geometric expressions like the volume are topological invariants. This rigidity is a property which we should expect for fermions. The usual matter is seen as dust matter (incompressible $p = 0$). The scaling behavior of the energy density ρ for dust matter is determined by the time-dependent scaling parameter a to be $\rho \sim a^{-3}$. So, if one represents matter by very small regions in the space equipped with a geometric structure then this scaling can be generated by an invariance of these small regions with respect to a rescaling. Mostow rigidity now singles out the hyperbolic geometry (and the hyperbolic 3-manifold as the corresponding small region) to have the correct behavior. All other geometries allow a scaling at least along one direction. The radiation (or interactions represented by bosons as gauge fields) has a scaling characteristics ($\rho \sim a^{-4}$) like these geometries. We will discuss the details especially the case of torus bundles (or connecting tubes) in section 8 more carefully.

For a general knot K (as splicing $K = K_1 \bowtie K_2$) we obtain

$$\begin{aligned} \int_{S^1 \times C(K)} R_K \sqrt{g_K} d^4x &= \int_{S^1 \times C(K_1)} R_K \sqrt{g_K} d^4x + \int_{S^1 \times T(K_1, K_2)} R_K \sqrt{g_K} d^4x + \\ &+ \int_{S^1 \times C(K_2)} R_K \sqrt{g_K} d^4x. \end{aligned}$$

The action over the knot complements can be written by (8) as integral over the mean curvature. Thus we obtain the contribution to the action for a general knot:

$$\begin{aligned} \int_{S^1 \times C(K)} R_K \sqrt{g_K} d^4x &= \lambda \cdot \text{vol}(S^1 \times S^3) - \sum_{n=1}^2 \int_{S^1 \times \partial N(K_n)} H_{\partial N(K)} \sqrt{g} d\theta d^2x + \\ &+ \int_{S^1 \times T(K_1, K_2)} R_K \sqrt{g_K} d^4x \quad (12) \end{aligned}$$

⁴ In 3 dimensions there are 8 geometric structures among them the spherical, Euclidean and hyperbolic geometry.

The constant in the first part can be interpreted as the *cosmological constant* Λ usually given in

$$\int_M \Lambda \sqrt{g} d^4x = \Lambda \cdot \text{vol}(M)$$

and the comparison to the action $\lambda \cdot \text{vol}(S^1 \times S^3)$ above gives

$$\Lambda = \lambda \cdot \frac{\text{vol}(S^1 \times S^3)}{\text{vol}(M)} . \quad (13)$$

And thus we see, that additionally to the usual terms of the Einstein-Hilbert-action one obtains two types of terms

$$\int_{S^1 \times \partial N(K_n)} H_{\partial N(K)} \sqrt{g} d\theta d^2x , \quad (14)$$

$$\int_{S^1 \times T(K_1, K_2)} R_K \sqrt{g_K} d^4x , \quad (15)$$

in (11). Both types of integrals describe the immersion of certain submanifolds into the 3-space. In the following we will show that these terms can be interpreted as the action of (fermionic) spinor fields as well as of (bosonic) gauge fields.

6 Dirac action

The action (14) above is completely determined by the knotted torus $\partial N(K) = K \times S^1$ and its mean curvature $H_{\partial N(K)}$. This knotted torus is an immersion of a torus $S^1 \times S^1$ into \mathbb{R}^3 . The well-known *Weierstrass representation* can be used to describe this immersion. As proved in [41, 30] there is an equivalent representation via spinors. This so-called *Spin representation* of a surface gives back an expression for $H_{\partial N(K)}$ and the Dirac equation as geometric condition on the immersion of the surface. As we will show below, the term (14) can be interpreted as Dirac action of a spinor field.

6.1 Weierstrass and spin representation of immersed submanifolds

In this subsection we describe the theory of immersions using spinors. The theory will be presented stepwise. We start with a toy model of an immersion of a surface into the 3-dimensional Euclidean space. Then we discuss how this map can be extended to an immersion of a 3-manifold into a 4-manifold.

Let $f : M^2 \rightarrow \mathbb{R}^3$ be a smooth map of a Riemannian surface with injective differential $df : TM^2 \rightarrow T\mathbb{R}^3$, i.e. an immersion. In the *Weierstrass representation* one expresses a *conformal minimal* immersion f in terms of a holomorphic function $g \in \Lambda^0$ and a holomorphic 1-form $\mu \in \Lambda^{1,0}$ as the integral

$$f = \text{Re} \left(\int (1 - g^2, i(1 + g^2), 2g)\mu \right) .$$

An immersion of M^2 is conformal if the induced metric g on M^2 has components

$$g_{zz} = 0 = g_{\bar{z}\bar{z}}, \quad g_{z\bar{z}} \neq 0$$

and it is minimal if the surface has minimal volume. Now we consider a spinor bundle S on M^2 (i.e. $TM^2 = S \otimes S$ as complex line bundles) and with the splitting

$$S = S^+ \oplus S^- = \Lambda^0 \oplus \Lambda^{1,0}$$

Therefore the pair (g, μ) can be considered as spinor field φ on M^2 . Then the Cauchy-Riemann equation for g and μ is equivalent to the Dirac equation $D\varphi = 0$. The generalization from a conformal minimal immersion to a conformal immersion was done by many authors (see the references in [30]) to show that the spinor φ now fulfills the Dirac equation

$$D\varphi = H\varphi \quad (16)$$

where H is the mean curvature (i.e. the trace of the second fundamental form). The minimal case is equivalent to the vanishing mean curvature $H = 0$ recovering the equation above. Friedrich [30] uncovered the relation between a spinor Φ on \mathbb{R}^3 and the spinor $\varphi = \Phi|_{M^2}$: if the spinor Φ fulfills the Dirac equation $D\Phi = 0$ then the restriction $\varphi = \Phi|_{M^2}$ fulfills equation (16) and $|\varphi|^2 = \text{const.}$ Therefore we obtain

$$H = \bar{\varphi} D\varphi \quad (17)$$

with $|\varphi|^2 = 1$.

After this exercise we are ready to consider the integral (14). Here we have an immersion of a torus $I : T^2 = S^1 \times S^1 \rightarrow \mathbb{R}^3$ with image the knotted torus $\text{im}(I) = T(K) = K \times S^1$ that is the boundary $\partial N(K)$ of $N(K)$. This immersion I can be defined by a spinor φ on T^2 fulfilling the Dirac equation

$$D\varphi = H\varphi \quad (18)$$

with $|\varphi|^2 = 1$ (or an arbitrary constant) (see Theorem 1 of [30]). As discussed above a spinor bundle over a surface splits into two sub-bundles $S = S^+ \oplus S^-$ with the corresponding splitting of the spinor φ in components

$$\varphi = \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix}$$

and we have the Dirac equation

$$D\varphi = \begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix} \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix} = H \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix}$$

with respect to the coordinates (z, \bar{z}) on T^2 .

In dimension 3, the spinor bundle has the same fiber dimension as the spinor bundle S (but without a splitting $S = S^+ \oplus S^-$ into two sub-bundles). Now we define the extended spinor ϕ over the 3-torus $T^3 = S^1 \times S^1 \times S^1$ via

the restriction $\phi|_{T^2} = \varphi$. The spinor ϕ is constant along the normal vector $\partial_N \phi = 0$ fulfilling the 3-dimensional Dirac equation

$$D^{3D} \phi = \begin{pmatrix} \partial_N & \partial_z \\ \partial_{\bar{z}} & -\partial_N \end{pmatrix} \phi = H \phi \quad (19)$$

induced from the Dirac equation (18) via restriction and where $|\phi|^2 = \text{const.}$ Especially one obtains for the mean curvature of the knotted 3-torus $T^3(K) = K \times S^1 \times S^1$ (up to a constant from $|\phi|^2$)

$$H = \bar{\phi} D^{3D} \phi. \quad (20)$$

6.2 The Dirac action in 3 dimensions and the 4-dimensional Dirac equation

By using the relation (20) above we obtain for the integral (14)

$$\int_{S^1 \times \partial N(K_n)} H_{\partial N(K)} \sqrt{g} d\theta d^2 x = \int_{S^1 \times \partial N(K)} \bar{\phi} D^{3D} \phi \sqrt{g} d\theta d^2 x \quad (21)$$

i.e. the Dirac action on the knotted 3-torus $S^1 \times \partial N(K) = K \times S^1 \times S^1 = T^3(K)$. But that is not the expected result, we obtain only a 3-dimensional Dirac action leaving us with the question to extend the action to four dimensions.

Let $\iota : T^3 \hookrightarrow M$ be an immersion of the 3-torus $\Sigma = T^3$ into the 4-manifold M with the normal vector \mathbf{N} . At this stage one can consider an arbitrary 3-manifold Σ instead of the 3-torus. The spin bundle S_M of the 4-manifold splits into two sub-bundles S_M^\pm where one subbundle, say S_M^+ , can be related to the spin bundle S_Σ of the 3-manifold. Then the spin bundles are related by $S_\Sigma = \iota^* S_M^+$ with the same relation $\phi = \iota_* \Phi$ for the spinors ($\phi \in \Gamma(S_\Sigma)$ and $\Phi \in \Gamma(S_M^+)$). Let $\nabla_X^M, \nabla_X^\Sigma$ be the covariant derivatives in the spin bundles along a vector field X as section of the bundle $T\Sigma$. Then we have the formula

$$\nabla_X^M(\Phi) = \nabla_X^\Sigma \phi - \frac{1}{2}(\nabla_X \mathbf{N}) \cdot \mathbf{N} \cdot \phi \quad (22)$$

with the obvious embedding $\phi \mapsto \begin{pmatrix} \phi \\ 0 \end{pmatrix} = \Phi$ of the spinor spaces. The expression $\nabla_X \mathbf{N}$ is the second fundamental form of the immersion where the trace $\text{tr}(\nabla_X \mathbf{N}) = 2H$ is related to the mean curvature H . Then from (22) one obtains a similar relation between the corresponding Dirac operators

$$D^M \Phi = D^{3D} \phi - H \phi \quad (23)$$

with the Dirac operator D^{3D} defined via (19). Together with equation (19) we obtain

$$D^M \Phi = 0 \quad (24)$$

i.e. Φ is a parallel spinor.

6.3 The extension to the 4-dimensional Dirac action

Above we obtained a relation (23) between a 3-dimensional spinor ϕ on the 3-manifold $\Sigma = T^3$ fulfilling a Dirac equation $D^\Sigma \phi = H\phi$ (determined by the immersion $\Sigma \rightarrow M$ into a 4-manifold M) and a 4-dimensional spinor Φ on a 4-manifold M with fixed chirality ($\in \Gamma(S_M^+)$ or $\in \Gamma(S_M^-)$) fulfilling the Dirac equation $D^M \Phi = 0$. At first we consider the variation

$$\delta \int_{S^1 \times \partial N(K)} \bar{\phi} D^{3D} \phi \sqrt{g} d\theta d^2 x = 0 \quad (25)$$

of the 3-dimensional action leading to the Dirac equations

$$D^{3D} \phi = 0 \quad D^{3D} \bar{\phi} = 0 \quad (26)$$

or to

$$H = 0,$$

a characterization of the immersion $S^1 \times \partial N(K)$ of the 3-torus T^3 with minimal mean curvature. This variation can be understood as a variation of the (conformal) immersion. In contrast, the extension of the spinor ϕ (as solution of (26)) to the 4-dimensional spinor Φ by using the embedding

$$\Phi = \begin{pmatrix} \phi \\ 0 \end{pmatrix} \quad (27)$$

can be only seen as immersion, if (and only if) the 4-dimensional Dirac equation

$$D^M \Phi = 0$$

on M is fulfilled (using relation (23)). This Dirac equation is obtained by varying the action

$$\delta \int_M \bar{\Phi} D^M \Phi \sqrt{g} d^4 x = 0 \quad (28)$$

Importantly, this variation has a different interpretation in contrast to varying the 3-dimensional action. Both variations look very similar. But in (28) we vary over smooth maps $\Sigma = T^3 \rightarrow M$ which are not conformal immersions (i.e. represented by spinors Φ with $D^M \Phi \neq 0$). Only the choice of the extremal action selects the conformal immersion among other smooth maps. Especially the spinor Φ (as solution of the 4-dimensional Dirac equation) is localized at the immersed 3-manifold Σ (with respect to the embedding (27)). The 3-manifold Σ moves along the normal vector (see the relation (22) between the covariant derivatives representing a parallel transport).

Therefore the 3-dimensional action (21) can be extended to the whole 4-manifold (but for a spinor Φ of fixed chirality). Especially we have a unique fermionic action on the manifold M . By combining the action (11) with (12)

and ignoring the action (15) of the connecting tubes, one obtains the pure fermionic action on M

$$\int_M (R + \bar{\Phi} D^M \Phi) \sqrt{g} d^4x \quad (29)$$

The action (29) is the usual Einstein-Hilbert action for a Dirac field Φ as source. What about the mass term? In our scheme there is one possible way to do it: using the constant length $|\Phi|^2 = \text{const.}$ of the spinor, we can introduce the scalar curvature R_Γ of an additional 3-manifold Γ with constant curvature coupled to the spinor. Then we obtain

$$\int_M \bar{\Phi} (D^M - m) \Phi \sqrt{g} d^4x$$

with $m = -R_\Gamma$ and $\Gamma \subset M$. But at the moment we have no idea how to generate realistic masses from this idea.

7 Gauge field action

Now we will discuss the second term (15)

$$\int_{S^1 \times T(K_1, K_2)} R_K \sqrt{g_K} d^4x$$

Using the product metric (5) and the splitting (6) as well the relation $\mathbf{K} = \text{const.}$ to rewrite this integral

$$\begin{aligned} S_{EH}(S^1 \times T(K_1, K_2)) &= \int_{S^1 \times T(K_1, K_2)} R_K \sqrt{g_K} d^4x \\ &= L_{S^1} \int_{T(K_1, K_2)} R_{(3)} \sqrt{h} N d^3x \end{aligned}$$

As shown by Witten [56, 57, 58], the action

$$\int_{T(K_1, K_2)} R_{(3)} \sqrt{h} N d^3x = L \cdot CS(T(K_1, K_2), A)$$

is related to the Chern-Simons action $CS(T(K_1, K_2), A)$ (defined in the appendix B) and we obtain for the action $S_{EH}(S^1 \times T(K_1, K_2))$

$$\int_{S^1 \times T(K_1, K_2)} R_K \sqrt{g_K} d^4x = L_{S^1} \cdot L \cdot CS(T(K_1, K_2), A) \quad (30)$$

with respect to the (Levi-Civita) connection A and the length L . For the 3-manifold $T(K_1, K_2)$, there is a 4-manifold M_T with $\partial M_T = T(K_1, K_2)$ (take

for instance $M_T = T(K_1, K_2) \times [0, 1] \subset T(K_1, K_2) \times S^1$. By using the Stokes theorem (see (35) in the appendix B) we obtain

$$S_{EH}(M_T) = \int_{M_T} \text{tr}(F \wedge F)$$

with the curvature $F = DA$, i.e. the action is the (topological) Pontrjagin class of the 4-manifold M_T . But $T(K_1, K_2)$ is a manifold with boundary and thus the variation of the action do not vanish. From the formal point of view, the curvature 2-form $F = DA$ is generated by a $SO(3, 1)$ connection A in the frame bundle, which can be lifted uniquely to a $SL(2, \mathbb{C})$ - (Spin-) connection. According to the Ambrose-Singer theorem, the components of the curvature tensor are determined by the values of holonomy which is in general a subgroup of $SL(2, \mathbb{C})$.

Thus we start with a suitable curvature 2-form $F = DA$ with values in the Lie algebra \mathfrak{g} of the Lie group G as subgroup of the $SL(2, \mathbb{C})$. The variation of the Chern-Simons action (30) gets flat connections $DA = 0$ as solutions. The flow of solutions $A(t)$ in $T(K_1, K_2) \times [0, 1]$ (parametrized by the variable t , the "time") between the flat connection $A(0)$ in $T(K_1, K_2) \times \{0\}$ to the flat connection $A(1)$ in $T(K_1, K_2) \times \{1\}$ will be given by the gradient flow equation (see [28] for instance)

$$\frac{d}{dt}A(t) = \pm * F(A) = \pm * DA \quad (31)$$

where the coordinate t is normal to $T(K_1, K_2)$. Therefore we are able to introduce a connection \tilde{A} in $T(K_1, K_2) \times [0, 1]$ so that the covariant derivative in t -direction agrees with $\partial/\partial t$. Then we have for the curvature $\tilde{F} = D\tilde{A}$ where the fourth component is given by $\tilde{F}_{4\mu} = d\tilde{A}_\mu/dt$. Thus we will get the instanton equation with (anti-)self-dual curvature

$$\tilde{F} = \pm * \tilde{F}.$$

It follows

$$S_{EH}([0, 1] \times T(K_1, K_2)) = \int_{T(K_1, K_2) \times [0, 1]} \text{tr}(\tilde{F} \wedge \tilde{F}) = \pm \int_{T(K_1, K_2) \times [0, 1]} \text{tr}(\tilde{F} \wedge * \tilde{F}),$$

i.e. the action of the gauge field. The whole procedure remains true for an extension of the "time", i.e.

$$S_{EH}(\mathbb{R} \times T(K_1, K_2)) = \pm \int_{T(K_1, K_2) \times \mathbb{R}} \text{tr}(\tilde{F} \wedge * \tilde{F}). \quad (32)$$

7.1 Extension to the 4-dimensional action

The gauge field action (32) is only defined along the tubes $T(K_1, K_2)$. For the extension of the action to the whole 4-manifold M , we need some non-trivial facts from the theory of 3-manifolds. At first the tubes $T(K_1, K_2)$ are so-called graph-manifolds. The decomposition of a general irreducible 3-manifold is given by the Thick-Thin decomposition: the thick part is a collection of hyperbolic 3-manifolds whereas the thin part is the set of graph manifolds. This decomposition is a result of the Mostow rigidity theorem [47], i.e. every conformal transformation (especially a scaling) of a hyperbolic manifold is an isometry. Therefore there exists a scaling which makes the thin part smaller but does not change the thick part. Of course the tubes $T(K_1, K_2)$ are contained in the thin part. We contract $T(K_1, K_2)$ to thin tubes connecting the thick parts. Conversely one also finds a scaling so that the thin part becomes large (but the thick part has the same size). Thus we can interpret the curvature \tilde{F} of the thin part as field located between the thick part. The thick part can be interpreted as fermions (see above). Then the action integral of the bosons can be written as

$$\int_{M \setminus \text{vol}(\text{fermion})} \text{tr}(\tilde{F} \wedge * \tilde{F})$$

i.e. like the integral (32) considered over $M \setminus \text{vol}(\text{fermions})$, the spacetime except the thick part (extended along the time axis). In the point-particle case we can extend this field to the whole manifold M . Then we obtain the gauge field action

$$\int_M \text{tr}(\tilde{F} \wedge * \tilde{F}) \quad (33)$$

Finally we summarize all terms for the whole action of the proposed geometrical model of matter (11)

$$S(M) = \int_M \left(R + \Lambda + \sum_n (\bar{\Phi} D^M \Phi)_n \right) \sqrt{g} d^4x + \int_M \text{tr}(\tilde{F} \wedge * \tilde{F}). \quad (34)$$

showing a combined Dirac-gauge-field coupled to the Einstein-Hilbert action.

8 Gauge group

In the last section we have seen, that the connecting tubes of the geometrical model can be interpreted as a gauge field. We will now discuss the possible gauge group of the obtained field. The gauge field in the action (33) has values in the Lie algebra of the maximal compact subgroup $SU(2)$ of $SL(2, \mathbb{C})$. But in the derivation of the action, we used the connecting tube $T(K_1, K_2)$ between two tori which is a cobordism. This cobordism $T(K_1, K_2)$ is also known as torus bundle (see [18] Theorem 1.15) which can be always decomposed into

three elementary pieces – *finite order*, *Dehn twist* and *Anosov map*⁵. The idea of this construction is very simple: one starts with two trivial cobordisms $T^2 \times [0, 1]$ and glue them together by using a diffeomorphism $T^2 \rightarrow T^2$. From the geometrical point of view, we have to distinguish between three different types of torus bundles. The three types of torus bundles are distinguished by the splitting of the tangent bundle:

- finite order (orders 2, 3, 4, 6): the tangent bundle is 3-dimensional
- Dehn-twist (left/right twist): the tangent bundle is a sum of a 2-dimensional and a 1-dimensional bundle
- Anosov: the tangent bundle is a sum of three 1-dimensional bundles.

Following Thurston’s geometrization program (see [55]), these three torus bundles are admitting a geometric structure, i.e. it has a metric of constant curvature. From the physical point of view, we have two types of diffeomorphisms: local and global. Any coordinate transformation can be described by an infinitesimal or local diffeomorphism. In contrast there are global diffeomorphisms like an orientation reversing diffeomorphism. Two diffeomorphisms not connected via a sequence of local diffeomorphism are part of different connecting components of the diffeomorphism group, i.e. the set of isotopy classes $\pi_0(Diff(M))$. Isotopy classes are important to understand the configuration space topology of general relativity (see Giulini [32]). Consider two different isotopy classes of a given 3-manifold together with a smooth, fixed metric on each class, say g_1 and g_2 , respectively. By definition, the two metrics g_1, g_2 cannot be connected via a smooth path, i.e. by a one-parameter subgroup of diffeomorphisms. Therefore two different isotopy classes represent two different physical situations, see [33] for the relation of isotopy classes to particle properties like spin. For example, let’s take the isotopy classes of a 4-manifold, which is related to exotic smoothness [51]. For that purpose we will consider these classes which in case of the torus bundles are given by

- finite order: 2 isotopy classes (= no/even twist or odd twist)
- Dehn-twist: 2 isotopy classes (= left or right Dehn twists)
- Anosov: 8 isotopy classes (= all possible orientations of the three line bundles forming the tangent bundle)

From the geometrical point of view, we can rearrange the scheme above:

- torus bundle with no/even twists: 1 isotopy class
- torus bundle with twist (Dehn twist or odd finite twist): 3 isotopy classes
- torus bundle with Anosov map: 8 isotopy classes

This information puts a starting point of the discussion how to derive the gauge group. Given a Lie group G with Lie algebra \mathfrak{g} . The rank of \mathfrak{g} is the dimension of the maximal abelian subalgebra, also called Cartan algebra. It is the same as the dimension of the maximal torus $T^n \subset G$. The curvature F of the gauge field takes values in the adjoint representation of the Lie algebra and the action $tr(F \wedge *F)$ forms an element of the Cartan subalgebra (the

⁵ The details of the construction is not important for the following discussion.

Casimir operator). But each isotopy class contributes to action and therefore we have to take the sum over the isotopy classes. Let t_a the generator in the adjoint representation, then we obtain for the Lie algebra part of the action $\text{tr}(F \wedge *F)$

- torus bundle with no twists: 1 isotopy class with t^2
- torus bundle with twist: 3 isotopy classes with $t_1^2 + t_2^2 + t_3^2$
- torus bundle with Anosov map: 8 isotopy classes with $\sum_{a=1}^8 t_a^2$.

The Lie algebra with one generator t corresponds uniquely to the Lie group $U(1)$ where the 3 generators t_1, t_2, t_3 form the Lie algebra of the $SU(2)$ group. Then the last case with 8 generators t_a have to correspond to the Lie algebra of the $SU(3)$ group. We remark the similarity with an idea from brane theory: n parallel branes (each decorated with an $U(1)$ theory) are described by an $U(n)$ gauge theory (see [34]). Finally we obtain the maximal group $U(1) \times SU(2) \times SU(3)$ as gauge group for all possible torus bundle (in the model: connecting tubes between the solid tori).

At the end we will speculate about the identification of the isotopy classes for the torus bundle with the vector bosons in the gauge field theory. Obviously the isotopy class of the torus bundle with no twist must be the photon. Then the isotopy class of the other bundle of finite order should be identified with the Z^0 boson and the two isotopy classes of the Dehn twist bundles are the W^\pm bosons. We remark that this scheme contains automatically the mixing between the photon and the Z^0 boson (the corresponding torus bundle are both of finite order). The isotopy classes of the Anosov map bundle have to correspond to the 8 gluons. We know that this approach left open many questions (like symmetry breaking, Higgs boson etc.).

9 Discussion

At the end of the paper we will summarize our assumptions and results. We started with a smooth 4-manifold M admitting an exotic smoothness structure M_K . This smoothness structure is constructed by using knot surgery. Then we discuss the general properties like the existence of a Lorentz metric and global hyperbolicity. Beginning with section 4 we considered the Einstein-Hilbert action on M_K and the decomposition

$$M = (M \setminus N(T^2)) \cup_{T^2} (S^1 \times (S^3 \setminus N(K)))$$

Because of the diffeomorphism invariance of the action, one can split the Einstein-Hilbert action like (2) leading to the relation (3). Then we were able to define an action over the knot complement to identify two contributions: knotted tori and connecting tubes between two tori.

1. knotted solid torus: As shown in section 6, a knotted solid torus can be described by a spinor so that the mean curvature is the Dirac action of this spinor. This action over the 3-dimensional boundary can be extended to the whole 4-manifold (29).

2. connecting tube: We discussed this case in section 7 using special properties of the tube as cobordism between two tori. Finally we obtained the Yang-Mills action (33).

We finish the paper with a conjecture about the gauge group. The connecting tubes can be identified with torus bundles which are classified. The three possible types of torus bundles were identified with three interactions to get the gauge group $U(1) \times SU(2) \times SU(3)$. Finally we can support the conjecture that exotic smoothness generates fermionic and bosonic fields.

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A Knot complement

Let $K : S^1 \rightarrow S^3$ be an embedding of the circle into the 3-sphere, i.e. a knot K . We define by $N(K) = D^2 \times K$ a thickened knot or a knotted solid torus. The knot complement $S^3 \setminus N(K)$ results in cutting $N(K)$ off from the 3-sphere S^3 . Then one obtains a 3-manifold with boundary $\partial(S^3 \setminus N(K)) = T^2$. The properties of the knot complement depend strongly on the properties of the knot K . So, the fundamental group $\pi_1(S^3 \setminus N(K))$ is also denoted as knot group. In contrast, the homology group $H_1(S^3 \setminus N(K)) = \mathbb{Z}$ don't depend on the knot.

B Chern-Simons invariant

Let P be a principal G bundle over the 4-manifold M with $\partial M \neq \emptyset$. Furthermore let A be a connection in P with the curvature

$$F_A = dA + A \wedge A$$

and Chern class

$$C_2 = \frac{1}{8\pi^2} \int_M \text{tr}(F_A \wedge F_A)$$

for the classification of the bundle P . By using the Stokes theorem we obtain

$$\int_M \text{tr}(F_A \wedge F_A) = \int_{\partial M} \text{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \quad (35)$$

with the Chern-Simons invariant

$$CS(\partial M, A) = \frac{1}{8\pi^2} \int_{\partial M} \text{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right). \quad (36)$$

Now we consider the gauge transformation $A \rightarrow g^{-1}Ag + g^{-1}dg$ and obtain

$$CS(\partial M, g^{-1}Ag + g^{-1}dg) = CS(\partial M, A) + k$$

with the winding number

$$k = \frac{1}{24\pi^2} \int_{\partial M} (g^{-1}dg)^3 \in \mathbb{Z}$$

of the map $g : M \rightarrow G$. Thus the expression

$$CS(\partial M, A) \bmod 1$$

is an invariant, the Chern-Simons invariant. Now we will calculate this invariant. For that purpose we consider the functional (36) and its first variation vanishes

$$\delta CS(\partial M, A) = 0$$

because of the topological invariance. Then one obtains the equation

$$dA + A \wedge A = 0,$$

i.e. the extrema of the functional are the connections of vanishing curvature. The set of these connections up to gauge transformations is equal to the set of homomorphisms $\pi_1(\partial M) \rightarrow SU(2)$ up to conjugation. Thus the calculation of the Chern-Simons invariant reduces to the representation theory of the fundamental group into $SU(2)$. In [25] the authors define a further invariant

$$\tau(\Sigma) = \min \{CS(\alpha) \mid \alpha : \pi_1(\Sigma) \rightarrow SU(2)\}$$

for the 3-manifold Σ . This invariant fulfills the relation

$$\tau(\Sigma) = \frac{1}{8\pi^2} \int_{\Sigma \times \mathbb{R}} \text{tr}(F_A \wedge F_A)$$

which is the minimum of the Yang-Mills action

$$\left| \frac{1}{8\pi^2} \int_{\Sigma \times \mathbb{R}} \text{tr}(F_A \wedge F_A) \right| \leq \frac{1}{8\pi^2} \int_{\Sigma \times \mathbb{R}} \text{tr}(F_A \wedge *F_A)$$

i.e. the solutions of the equation $F_A = \pm *F_A$. Thus the invariant $\tau(\Sigma)$ of Σ corresponds to the self-dual and anti-self-dual solutions on $\Sigma \times \mathbb{R}$, respectively. Or the invariant $\tau(\Sigma)$ is the Chern-Simons invariant for the Levi-Civita connection.

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